JEE Mains 2020

7th Jan Shift 1 Paper – Mathematics

1. If $f(a+b+1-x)=f(x)$, for all x, where a and b are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x\big(f(x)+f(x+1)\big)\,dx$ is equal to:

Sol. $f(a+b+1-x)=f(x)$

Put $x=1+x$

$f(a+b-x)=f(x+1)$

$$I=\frac{1}{a+b}\int_a^b [x(f(x)+f(x+1))]\,dx \qquad \text{i)}$$

$$I=\frac{1}{a+b}\int_a^b [(a+b-x)(f(a+b-x)+f(a+b-x+1))]\,dx$$

$$I=\frac{1}{a+b}\int_a^b [(a+b-x)(f(x+1)+f(x))]\,dx \qquad \text{ii)}$$

Adding i and ii

$$I=\frac{1}{a+b}\int_a^b [(a+b)(f(x+1)+f(x))]\ dx$$

$$I=\int_a^b [(f(x+1)+f(x))]\,dx$$

$$2I=\int_a^b f(x+1)\,dx+\int_a^b f(x)\,dx$$

$$2I=\int_a^b f(a+b+1-x)\,dx+\int_a^b f(x)\,dx$$

$$2I=2\int_a^b f(x)\,dx$$

2. Let α and β be two real roots of the equation $(k+1)\tan^2(x)-\sqrt{2}\lambda\tan(x)=(1-K)$, where $K\neq 1$ and λ are real numbers. If $\tan^{-1}(\alpha+\beta)=50$. Then value of λ is :

Sol. $\tan\alpha+\tan\beta=\frac{\sqrt{2}\lambda}{1+k}$ given

$\tan\alpha\times\tan\beta=\frac{(k-1)}{1+k}$

Since $\tan\alpha$ & $\tan\beta$ are the roots

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=50$$

$$\Rightarrow \frac{\frac{\sqrt{2}\lambda}{1+k}}{1-\frac{k-1}{k+1}}=50$$

$\lambda=10$

3. A total number of 6 digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 appear is:

Sol. $5\times\frac{6!}{2!}$

4. If $y = mx + 4$ is a tangent to both the parabola, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to:

Sol. $y = mx - am^2$

$y = \frac{x}{4} + 4$

$y = 4x, y = mx + \frac{1}{m} \Rightarrow m = \frac{1}{4}$

Also, $x^2 = 2by$

$y = mx - \frac{bm^2}{2}$

$y = \frac{x}{4} + 4$

$\Rightarrow \frac{bm^2}{2} = -4$

since, $m = \frac{1}{4}$

Therefore, $b = -128$

5. Let α be a root of the equation $x^2 + x + 1 = 0$ and the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$, then the matrix A^{31} is equal to:

Sol. Given $x^2 + x + 1 = 0$

$\Rightarrow \alpha$ is the root of the equation $\alpha = \omega, \omega^2$

$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$

$A^2 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & \omega^2 & \omega \end{bmatrix}$

$A^4 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$

$A^4 = I$

$\therefore A^{31} = A^{28} \cdot A^3 = A^3$

6. If $y = y(x)$ is the solution of the differential equation, $e^y\left(\frac{dy}{dx} - 1\right) - e^x$, such that $y(0) = 0$ then $y(1)$ is equal to:

Sol. $\frac{d}{dx}\left(\frac{e^y}{e^x}\right) = \frac{e^y . e^x \frac{dy}{dx} - e^x . e^y}{(e^x)^2}$

$\int \frac{d}{dx}\left(\frac{e^y}{e^x}\right) = \int 1 \cdot dx$

$\left(\frac{e^y}{e^x}\right) = x$

$(e^y) = xe^x + c$, at $y(0) = 0$

$c = 1$

$y(1) = 1 + \ln 2$

7. If $y(\alpha) = \sqrt{2\left(\frac{\tan\alpha + \cot\alpha}{1 + \tan^2\alpha}\right) + \frac{1}{\sin^2\alpha}}, \alpha \in \left[\frac{3\pi}{4}, \pi\right]$ then $\frac{dy}{dx}$ at $\alpha = \frac{5\pi}{4}$ is :

Sol. $y(x) = \sqrt{2\frac{\tan x + \cot x}{1 + \tan^2 x} + \frac{1}{\sin^2 x}}$

$= \sqrt{2\frac{\frac{1}{\sin x \cos x}}{a + \frac{\sin^2 x}{\cos^2 x}} + \frac{1}{\sin^2 x}} = \sqrt{2\frac{\cos x}{\sin x} + \frac{1}{\sin^2 x}}$

$= \sqrt{\frac{2\cos x \sin x + 1}{\sin^2 x}} = \sqrt{\frac{(\cos x + \sin x + 1}{\sin^2 x}}$

$= \frac{(\cos x + \sin x)}{\sin x}$

$|1 + \cot x| = \frac{d}{dx}(-1 - \cot x) = \csc^2 x$

Put $x = \frac{5\pi}{6}$,

We get 4

8. Let the function $f: [-7, \infty] \to R$ be continuous on $[-7,0]$ and differentiable on $(-7,0)$. If $f(-7) = -3$ and $f'(x) \leq 2$ for all $x \in (-7,0)$ then for all such functions f, $f(-1) + f(0)$ lies in the interval:

Sol. Directly use :

$$\frac{f(-1) - f(-7)}{-1+7} = f'(c) \; \& \frac{f(0) - f(-7)}{-1+7} = f'(c)$$

$$\frac{f(-1) - f(-7)}{-1+7} = f'(2) \; \& \frac{f(0) - f(-7)}{0+7} = f'(2)$$

We get, $f(-1) \leq 9 \; \& \; f(0) \leq 11$

$f(-1) + f(0) \leq 20$

9. A vector $\vec{a} = \alpha\hat{\imath} + 2\hat{\jmath} + \beta\hat{k}, (\alpha, \beta \in R)$ lies in the plane of the vectors $\vec{b} = \hat{\imath} + \hat{\jmath}$ and $\vec{c} = \hat{\imath} - \hat{\jmath} + 4\,\hat{k}$. If $\vec{a}$ bisects the anglel between $\vec{b}$ and $\vec{c}$. Then.

Sol. Given $\vec{a} = \alpha\hat{\imath} + 2\hat{\jmath} + \beta\hat{k}, \vec{b} = \hat{\imath} + \hat{\jmath}, \vec{c} = \hat{\imath} - \hat{\jmath} + 4\hat{k}$ are coplanar.

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$

$$\Rightarrow \begin{bmatrix} \alpha & 2 & 3 \\ 1 & 1 & 0 \\ 1 & -1 & 4 \end{bmatrix} = 0$$

$\Rightarrow 4\alpha - 8 - 2\beta = 0$

Also it is given that $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$

$$\Rightarrow \frac{\alpha + 2}{|\vec{a}||\vec{b}|} = \frac{\alpha - 2 + 4\beta}{|\vec{a}||\vec{c}|}$$

$(\alpha, \beta) = (4, 4)$

10. If distance between the foci of an ellipse is 6 and the distance between directrices is 12, then the length of its latus rectum is:

Sol. Given $2ae = 6$

$\therefore ae = 3$ (1)

Also $\frac{2a}{e} = 126$

$\therefore \frac{a}{e} = 63$ (2)

From (1)and (2)

$e = \frac{1}{\sqrt{2}}, a = 3\sqrt{2}$

Since, $b^2 = a^2(1 - e^2)$

Substitute the values of 'e' and 'a' in above equation.

$\Rightarrow b^2 = 9$

$\therefore b = \pm 3$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{3\sqrt{2}} = 3\sqrt{2}$

11. The greatest positive integer k, for which $49^k + 1$ is factor of the sum $49^{125} + 49^{124} + \cdots 49^2 + 49 + 1$, is :

Sol. $1 + 49 + 49^2 + \cdots 49^{125}$

$$\Rightarrow \frac{a(r^{n+1} - 1)}{r - 1} \Rightarrow \frac{1(49^{126} - 1)}{48}$$

$$\Rightarrow \frac{(49^{63} - 1)(49^{63} + 1)}{48}$$

Hence, $k = 63$

12. Let P be the plane passing through the pts. (2,1,0), (4,1,1) and (5,0,1) and R be any point (2,1,6). Then the image of R in the plane P is:

Sol. Points of plane $p(2,1,0), (4,1,1)$ *and* $(5,0,1)$ and point $R(2,1,6)$.

Then the image of R in the plane P is:

$$\begin{vmatrix} x-2 & y-1 & z \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix} = (x-2)[0+1] - (y-1)[2-3] + z[-2]$$

$\Rightarrow x - 2 + y - 1 - 2z = 0,$

Plane, $x + y - 2z = 3$

Image $= (1,1,-2)$

13. If the system of linear equation

$2x + 2ay + az = 0$

$2x + 3ay + bz = 0$

$2x + 4ay + cz = 0$

Where $a, b, c \in R$ are non-zero and distinct; has a non-zero solution then:

Sol. For non zero solutions $D = 0$

$$D = \begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0$$

$C_2 \rightarrow C_2 - 2C_3$

$$D = \begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 2 & 2c & c \end{vmatrix} = 0$$

On solving $-bc + 2ac - ab = 0$

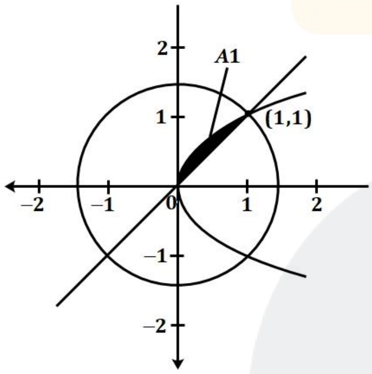
$$\frac{1}{1} + \frac{1}{c} = \frac{2}{b}$$

14. The area of the region enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is:

Sol. $x^2 + y^2 = 2 \Rightarrow r = \sqrt{2}$

$y^2 = x$

$y = x$



15. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to:

Sol.

p	q	$(p \rightarrow q) \wedge (q \rightarrow p)$
F	F	T
F	T	T
T	F	F
T	T	T

Which is equal to $\sim p$

16. An unbiased coin is tossed 5 *times*. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for $k = 3, 4, 5$ .Otherwise X takes the value -1. Then the expected value of X is :

Sol. $E(X) = 3 \times \left(\frac{1}{2}\right)^5 \times 3 + 2 \times 4 \times \left(\frac{1}{2}\right)^5 + 5 \times \left(\frac{1}{2}\right)^5 - 18 \times \left(\frac{1}{2}\right)^5$

$$= \left(\frac{1}{2}\right)^5 [9 + 8 + 5] - \frac{18}{32} = \left|\frac{4}{32}\right| = \frac{1}{8}$$

17. Let $x^k + y^k = a^k, (a, k > 0)$ and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{1/3} = 0$, then '$k$' is:

Sol. $x^k + y^k = a^k \quad (a, k > 0)$

$$\frac{dy}{dx} + \left(\frac{y}{x}\right)^{1/3} = 0 \Rightarrow kx^{k-1} + ky^{k-1} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0$$

$k - 1 = -1/3$

$k = 2/3$

18. If $Re\left[\frac{z-1}{2z+i}\right] = 1$, where $z = x + iy$, then the point (x, y) lies on a:

Sol. $Re\left(\frac{z-1}{2z+i}\right) = 1$

Put $z = x + iy$

$$Re\left(\frac{z-1}{2z+i}\right) = \frac{(x-1)2x + y(2y+1)}{4x^2 + 4y^2 + 1 + 4y} = 1$$

$$x^2 + y^2 + \left(\frac{3}{2}\right)y + x + \frac{1}{2} = 0$$

$$g = -\frac{1}{2}, f = -3/4$$

Radius $= \sqrt{5}/4$

19. If $g(x) = x^2 + x - 1$ and $gof\ (x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to:

Sol. $g(x) = x^2 + x - 1$

$gof(x) = 4x^2 - 10x + 5$

$f(x) = ax + b$

$gof(x) = (ax + b)^2 + ax + b - 1$

$gof(x) = a^2x^2 + b^2 + 2axb + ax + b - 1 = 4x^2 - 10x + 5$

$\Rightarrow a^2x^2 + x(2ab + a) + b^2 + b - 1 = 4x^2 - 10x + 5$

Therefore, $a^2 = 4, 2ab + a = -10, b^2 + b - 1 = 5$

Solving the above equations,

$a = 2, b = -3$

Therefore, $f(x) = 2x - 3$

$$f\left(\frac{5}{4}\right) = 2 \times \frac{5}{4} - 3 = \frac{-1}{2}$$

20. Five numbers are in A.P. whose sum is 25 and product is 2520. If one of these five numbers is $.\frac{-1}{2}$, then the greatest number amongst them is:

Sol: Five terms are in A.P.

Let terms are $a - 2d, a - d, a, a + d, a + 2d$

Sum of the terms

$5a = 25$

$a = 5$

Product of the terms

$(a - 2d) \times (a - d) \times a \times (a + d) \times (a + 2d) = 2502$

$d = \pm 1, \pm 11/2$

$d = \frac{11}{2}$ and $-\frac{11}{2}$ satisfies the condition

So, $a + 2d = 16$

21. Let $A(1,0), B(6,2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertices of a triangle ABC. If P is a point inside the ΔABC such that triangles $\Delta APC, \Delta APB$ ad ΔBPC have equal area. Then the length of the line segment PQ, where Q is the point $\left(-\frac{7}{6}, -\frac{1}{3}\right)$:

Sol. $A(1,0)\ B(6,2)\ C\left(\frac{3}{2}, 6\right)$

Point P is the centroid of triangle ABC

$P(17/6, 8/3)$

Distance between PQ is 5

22. If the variance of the first 'n' natural numbers is 10 and the variance of the first 'm' even natural numbers is 16, then $m + n$ is equal to:

Sol. $\sigma_n = \frac{n^2-1}{12} = 10$

$$\sigma_{m(\text{even})} = \frac{m^2 - 1}{3} = 16$$

Therefore $n = 11$

$m = 7$

$n + m = 18$

23. $\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{\frac{-x}{2}}-3^{1-x}}$ is equal to :

Sol. $\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{\frac{-x}{2}}-3^{1-x}}$

$$\Rightarrow \lim_{x\to 2}\frac{3^x+\frac{3^3}{3^x}-12}{\frac{1}{3^{x/2}}-\frac{3}{3^x}}$$

$$\Rightarrow \lim_{x\to 2}\frac{3^{2x}+3^3-3^x\cdot 12}{3^{\frac{x}{2}}-3}$$

$$\Rightarrow \frac{2\ln 3\cdot 9^x-4\ln(3)\cdot 3^{x+1}}{\frac{1}{2}\ln(3)e^{\frac{x}{2}\ln\left(\frac{3x}{2}\right)}}$$

$$\Rightarrow \frac{4\cdot 9^x-8\cdot 3^{x+1}}{3x}$$

For, put $x=2$ in the above equation, to get

$$\Rightarrow \frac{4\cdot 9^2-8\cdot 3^{2+1}}{3\times 2}=18$$

24. If the sum of the coefficients of all even powers of 'x' in the product $(1+x+x^2+\cdots x^2n)(1-x+x^2-x^3+\cdots x^2n)$ is 61 then 'n' is equal to:

Sol. Let $[(1-x+x^2\ldots)(1+x+x^2\ldots)]=a_0+a_1x+a_2x^2+\cdots$

Put $x=1$

$1(2n+1)=a_0+a_1+a_2+\cdots a_{2n}$ (1)

Put $x=-1$

$(2n+1)\times 1=a_0-a_1+a_2+\cdots a_{2n}$ (2)

From (1) +(2)

$4n+2=2(a_0+a_2+\cdots)$

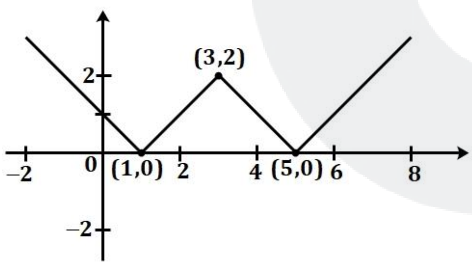
$=2\times 61$

$\Rightarrow 2n+1=61\Rightarrow n=30$

25. Let S be the set of points where the function $f(x)=|2-|x-3||\cdot x\in R$, is not differentiable. Then $\sum_{x\in S}f(f(x))$ is equal to:

Sol. $f(x)=|2-|x-3||$

$f(f(x))=\left|2-\left||2-|x-3||-3\right|\right|$



$S=\{(1,0),(3,2),(5,0)\}$

$$\sum_{x\in S}f(f(x))=\sum_{x\in S}\left|2-\left||2-|x-3||-3\right|\right|$$

$=3$