Avanti

JEE Mains 2020

8th Jan Shift 1 Paper – Mathematics

1. Let $\int \frac{\cos x dx}{\sin^3 x(1+\sin^6 x)^{\frac{2}{3}}} = f(x)(1+\sin^6 x)^{\frac{1}{\lambda}} + c$ then find value of $\lambda f\left(\frac{\pi}{3}\right)$

A) 4 B) -2 C) 8 D) -4

Ans. B

Sol. $\sin x = t$

$\cos x\, dx = dx$

$$I = \int \frac{dt}{t^3(1+t^6)^{\frac{2}{3}}} = \int \frac{dt}{t^7\left(1+\frac{1}{t^6}\right)^{\frac{2}{3}}}$$

Put $1+\frac{1}{t^6} = r^3 \Rightarrow \frac{dt}{t^7} = \frac{-1}{2}r^2 dr$

$$-\frac{1}{2}\int \frac{r^2 dr}{r^2} = -\frac{1}{2}r + c = -\frac{1}{2}\left(\frac{\sin^6 x + 1}{\sin^6 x}\right)^{\frac{1}{3}} + c = -\frac{1}{2\sin^2 x}(1+\sin^6 x)^{\frac{1}{3}} + c$$

$f(x) = -\frac{1}{2}\text{cosec}^2 x$ and $\lambda = 3$

$\lambda f\left(\frac{\pi}{3}\right) = -2$

2. If $y(x)$ is a solution of differential equation $\sqrt{1-x^2}\frac{dy}{dx} + \sqrt{1-y^2} = 0$, such that $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$, then

A) $y\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$ B) $y\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2}$ C) $y\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ D) $y\left(\frac{1}{2}\right) = \frac{1}{2}$

Ans. C

Sol. $\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0 \Rightarrow \sin^{-1} y + \sin^{-1} x = c$

At $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2} \Rightarrow c = \frac{\pi}{2} \Rightarrow \sin^{-1} y = \cos^{-1} x$

Hence $y\left(\frac{1}{\sqrt{2}}\right) = \sin\left(\cos^{-1}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

3. $\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}}$ is equal to

A) e^{-2} B) e^2 C) $e^{2/7}$ D) $e^{3/7}$

Ans. A

Sol. Let $L = \lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x^2}\left\{\frac{3x^2+2}{7x^2+2}-1\right\}} = e^{\lim_{x\to 0}\frac{1}{x^2}\left\{\frac{-4x^2}{7x^2+2}\right\}} = e^{\frac{-4}{2}} = e^{-2}$

4. In a bag there are 5 red balls, 3 white balls and 4 black balls. Four balls are drawn from the bag. Find the number of ways of the which at most 3 red balls are selected

A) 450 B) 360 C) 490 D) 510

Ans. C

Sol. Number of ways $= \underbrace{{}^7C_4}_{\text{0 Red}} + \underbrace{{}^5C_1 \cdot {}^7C_3}_{\text{1 Red}} + \underbrace{{}^5C_2 \cdot {}^7C_2}_{\text{2 Red}} + \underbrace{{}^5C_3 \cdot {}^7C_1}_{\text{3 Red}} = 35 + 175 + 210 + 70 = 490$

5. Let $f(x) = \{(\sin(\tan^{-1} x) + \sin(\cot^{-1} x))\}^2 - 1$ where $|x| > 1$ and $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(\sin^{-1} f(x))$.

If $y(\sqrt{3}) = \frac{\pi}{6}$ then $y(-\sqrt{3}) =$

A) $\frac{5\pi}{6}$ B) $\frac{-\pi}{6}$ C) $\frac{\pi}{3}$ D) $\frac{2\pi}{3}$

Ans. B

Sol. $2y = \sin^{-1} f(x) + C = \sin^{-1}(\sin(2\tan^{-1} x)) + C$

$\Rightarrow 2\left(\frac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) + C$

$\frac{\pi}{3} = \frac{\pi}{3} + C$

$\therefore C = 0$

for $x = -\sqrt{3}, 2y = \sin^{-1}\left(\sin\left(\frac{-2\pi}{6}\right)\right) + 0 \Rightarrow 2y = \frac{-\pi}{3}$

$\left(y = \frac{-\pi}{6}\right)$

6. If $2^{1-x} + 2^{1+x}, f(x), 3^x + 3^{-x}$ are in A.P. then minimum value of $f(x)$ is

A) 1 B) 2 C) 3 D) 4

Ans. C

Sol. $f(x) = \left(\frac{2^{1-x}+2^{1+x}+3^x+3^{-x}}{2}\right)$

Using AM $\geq$ GM

$f(x) \geq 3$

7. Which of the following is tautology

A) $\left(p \vee (p \to q)\right) \to q$ B) $q \to p \wedge (p \to q)$

C) $p \vee (p \wedge q)$ D) $(p \wedge (p \vee q))$

Ans. A

Sol.

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q) \to q)$	$q \to p \wedge (p \to q)$	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T	T	T	T	T	T	T
T	F	F	F	T	T	F	T	T	T
F	T	T	F	T	F	F	F	T	F
F	F	T	F	T	T	F	F	F	F

8. A is a 3×3 matrix whose elements are from the set $\{-1, 0, 1\}$. Find the number of matrices A such that $tr(AA^T) = 3$. Where $tr(A)$ is sum of diagonal elements of matrix A.

A) 572 B) 612 C) 672 D) 682

Ans. C

Sol. Let $A = [a_{ij}]_{3\times3}$

$tr(AA^T) = 3$

$a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{21}^2 + ... + a_{33}^2 = 3$

Possible cases

$$\left.\begin{array}{ll} 0,0,0,0,0,0,1,1,1 & \to 1 \\ 0,0,0,0,0,0,-1,-1,-1 & \to 1 \\ 0,0,0,0,0,0,1,1,-1 & \to 3 \\ 0,0,0,0,0,0,-1,1,-1 & \to 3 \end{array}\right\} {}^9C_6 \times 8 = 84 \times 8 = 672$$

9. Mean and standard deviations of 10 observations are 20 and 2 respectively. If $p = (p \neq 0)$ is multiplied to each observation and then $q (q \neq 0)$ is subtracted then new mean and standard deviation becomes half of original value. Then find q

A) -10 B) -20 C) -5 D) 10

Ans. B

Sol. If each observation is multiplied with p & then q is subtracted

New mean $\bar{x}_1 = p\bar{x} - q$

$\Rightarrow 10 = p(20) - q$ (1)

and new standard deviations

$\sigma_2 = |p|\sigma_1 \Rightarrow 1 = |p|$ (2)

$\Rightarrow |p| = \frac{1}{2}$

$\Rightarrow p = \pm\frac{1}{2}$

If $p = \frac{1}{2}$

then $q = 0$ (from equation (1))

If $p = -\frac{1}{2}$

$q = -20$

10. If maximum value of $^{19}C_p$ is a, $^{20}C_q$ is b, $^{21}C_r$ is c, then relation between a, b, c is

A) $\frac{a}{11} = \frac{b}{22} = \frac{c}{42}$ B) $\frac{a}{22} = \frac{b}{11} = \frac{c}{42}$ C) $\frac{a}{22} = \frac{b}{42} = \frac{c}{11}$ D) $\frac{a}{21} = \frac{b}{11} = \frac{c}{22}$

Ans. (A)

Sol. We know ^nC_r is max at middle term

$a = {^{19}C_p} = {^{19}C_{10}} = {^{19}C_9}$

$b = {^{20}C_q} = {^{20}C_{10}}$

$c = {^{21}C_6} = {^{21}C_{10}} = {^{21}C_{11}}$

$$\frac{a}{^{19}C_9} = \frac{b}{\frac{20}{10} \cdot {^{19}C_9}} = \frac{c}{\frac{21}{11} \cdot \frac{20}{10} {^{19}C_9}}$$

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{42/11}$$

11. Let $P(A) = \frac{1}{3}, P(B) = \frac{1}{6}$ where A and B are independent events then

A) $P\left(\frac{A}{B}\right) = \frac{1}{6}$ B) $P\left(\frac{A}{B'}\right) = \frac{1}{3}$ C) $P\left(\frac{A}{B'}\right) = \frac{2}{3}$ D) $P\left(\frac{A}{B}\right) = \frac{5}{6}$

Ans. B

Sol. A and B are independent events so

$$P\left(\frac{A}{B'}\right) = \frac{1}{3}$$

12. Let $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$ then inverse of $f(x)$ is

A) $\frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right)$ B) $\frac{1}{2}\log_8\left(\frac{1-x}{1+x}\right)$ C) $\frac{1}{4}\log_8\left(\frac{1-x}{1+x}\right)$ D) $\frac{1}{2}\log_8\left(\frac{1+x}{1-x}\right)$

Ans. A

Sol. $y = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$

$$\frac{1+y}{1-y} = \frac{8^{2x}}{8^{-2x}}$$

$$8^{4x} = \frac{1+y}{1-y}$$

$$4x = \log_8\left(\frac{1+y}{1-y}\right)$$

$$x = \frac{1}{4}\log_8\left(\frac{1+y}{1-y}\right)$$

$$f^{-1}(x) = \frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right)$$

13. Roots of the equation $x^2 + bx + 45 = 0, b \in R$ lie on the curve $|z+1| = 2\sqrt{10}$, where z is a complex number then

A) $b^2 + b = 12$ B) $b^2 - b = 30$ C) $b^2 - b = 36$ D) $b^2 + b = 30$

Ans. B

Sol. Let $z = \alpha \pm i\beta$ be roots of the equation

So $2\alpha = -b$ and $\alpha^2 + \beta^2 = 45, (\alpha+1)^2 + \beta^2 = 40$

So $(\alpha+1)^2 - \alpha^2 = -5$

$\Rightarrow 2\alpha + 1 = -5 \Rightarrow 2\alpha = -6$

So $b = 6$

Hence $b^2 - b = 30$

14. For $f(x) = \ln\left(\frac{x^2+\alpha}{7x}\right)$. Rolle's theorem is applicable on $[3, 4]$, the value of $f''(c)$ is equal to

A) $\frac{1}{12}$ B) $\frac{-1}{12}$ C) $\frac{1}{6}$ D) $\frac{-1}{6}$

Ans. A

Sol. $f(3) = f(4) \Rightarrow \alpha = 12$

$$f'(x) = \frac{x^2 - 12}{x(x^2+12)}$$

$\therefore f'(c) = 0$

$\therefore c = \sqrt{12}$

$$\therefore f''(c) = \frac{1}{12}$$

15. Let $f(x) = x\cos^{-1}(\sin(-|x|))\,, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ then

A) $f'(0) = -\frac{\pi}{2}$

B) $f'(x)$ is not defined at $x = 0$

C) $f'(x)$ is increasing in $\left(\frac{-\pi}{2}, 0\right)$ and $f'(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

D) $f'(x)$ is decreasing in $\left(\frac{-\pi}{2}, 0\right)$ and $f'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$

Ans. D

Sol. $f'(x) = x(\pi - \cos^{-1}(\sin|x|))$

$$= x\left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(\sin|x|)\right)\right) = x\left(\frac{\pi}{2} + |x|\right)$$

$$f(x) = \begin{cases} x\left(\frac{\pi}{2} + x\right) & x \geq 0 \\ x\left(\frac{\pi}{2} - x\right) & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{\pi}{2} + 2x & x \geq 0 \\ \frac{\pi}{2} - 2x & x < 0 \end{cases}$$

$f'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{-\pi}{2}, 0\right)$

16. Let P be a point on $x^2 = 4y$. The segment joining $A(0, -1)$ and P is divided by point Q in the ratio $1 : 2$, then locus of point Q is

A) $9x^2 = 3y + 2$ B) $9x^2 = 12y + 8$ C) $9y^2 = 12x + 8$ D) $9y^2 = 3x + 2$

Ans. B

Sol. Let point P be $(2t, t^2)$ and Q be (h, k).

$$h = \frac{2t}{3}, k = \frac{-2 + t^2}{3}$$

Hence locus is $3k + 2 = \left(\frac{3h}{2}\right)^2 \Rightarrow 9x^2 = 12y + 8$

17. Ellipse $2x^2 + y^2 = 1$ and $y = mx$ meet a point A in first quadrant. Normal to the ellipse at P meets x-axis at $\left(-\frac{1}{3\sqrt{2}}, 0\right)$ and y-axis at $(0, \beta)$, then $|\beta|$ is

A) $\frac{2}{\sqrt{3}}$ B) $\frac{2\sqrt{2}}{3}$ C) $\frac{\sqrt{2}}{3}$ D) $\frac{2}{3}$

Ans. C

Sol. Equation of normal at P is $\frac{x}{2x_1} - \frac{y}{y_1} = -\frac{1}{2}$

It passes through $\left(-\frac{1}{3\sqrt{2}}, 0\right) \Rightarrow \frac{-1}{6\sqrt{2}x_1} = -\frac{1}{2} \Rightarrow x_1 = \frac{1}{3\sqrt{2}}$

So $y_1 = \frac{2\sqrt{2}}{3}$ (as P lies in Ist quadrant)

So $\beta = \frac{y_1}{2} = \frac{\sqrt{2}}{3}$

18. If $y^2 = ax$ and $x^2 = ay$ intersect at A & B. Area bounded by both curves is bisected by line $x = b$ (given $a > 1 > 0$). Area of triangle formed by line $AB, x = b$ and x-axis is $\frac{1}{2}$. Then

A) $a^6 - 12a^3 - 4 = 0$ B) $a^6 + 12a^3 - 4 = 0$

C) $a^6 - 12a^3 + 4 = 0$ D) $a^6 + 12a^3 + 4 = 0$

Ans. C

Sol. $\int_0^b \left(\sqrt{a}x^{\frac{1}{2}} - \frac{x^2}{a}\right) dx = \frac{a^2}{6}$

$\Rightarrow \frac{2}{3}\sqrt{a}b^{\frac{3}{2}} - \frac{b^3}{3a} = \frac{a^2}{6}$ (1)

also area of $\Delta OQR = \frac{1}{2}$

$\frac{1}{2}b^2 = \frac{1}{2} \Rightarrow b = 1$

Put in (1)

$\Rightarrow 4a\sqrt{a} - 2 = a^3$

$\Rightarrow a^6 + 4a^3 + 4 = 16a^3$

$\Rightarrow a^6 - 12a^3 + 4 = 0$

19. Let ABC is a triangle whose vertices are $A(1,-1), B(0,2), C(x',y')$ and area of ΔABC is 5 and $C(x',y')$ lie on $3x+y-4\lambda=0$, then

A) $\lambda=3$ B) $\lambda=-3$ C) $\lambda=4$ D) $\lambda=2$

Ans. A

Sol. $D=\frac{1}{2}\begin{vmatrix}0 & 2 & 1\\ 1 & -1 & 1\\ x' & y' & 1\end{vmatrix}$

$-2(1-x')+(y'+x')=\pm 10$

$-2+2x'+y'+x'=\pm 10$

$3x'+y'=12$ or $3x'+y'=-8$

$\lambda=3,-2$

20. The system of equation $\begin{matrix}3x+4y+5z=\mu\\ x+2y+3z=1\\ 4x+4y+4z=\delta\end{matrix}$ is inconsistent, then (δ,μ) can be

A) $(4,6)$ B) $(3,4)$ C) $(4,3)$ D) $(1,0)$

Ans. C

Sol. Note $D=\begin{vmatrix}3 & 4 & 5\\ 1 & 2 & 3\\ 4 & 4 & 4\end{vmatrix}(R_3\to R_3-2R_1+3R_2)=\begin{vmatrix}3 & 4 & 5\\ 1 & 2 & 3\\ 0 & 0 & 0\end{vmatrix}=0$

Now let $P_3\equiv 4x+4y+4z-\delta=0$. If the system has solution it will have infinite solution, so $P_3\equiv \alpha P_1+\beta P_2$

Hence $3\alpha+\beta=4$ & $4\alpha+2\beta=4 \Rightarrow \alpha=2$ & $\beta=-2$

So for infinite solution $2\mu-2=\delta \Rightarrow$ for $2\mu\neq\delta+2$

system inconsistent

21. Shortest distance between the lines $\frac{x-3}{1}=\frac{y-8}{4}=\frac{z-3}{22}, \frac{x+3}{1}=\frac{y+7}{1}=\frac{z-6}{7}$ is

A) $3\sqrt{30}$ B) $2\sqrt{30}$ C) $\sqrt{30}$ D) $4\sqrt{30}$

Ans. A

Sol. $\overrightarrow{AB}=6\hat{i}+15\hat{j}+3\hat{k}$

$\vec{p}=\hat{i}+4\hat{j}+22\hat{k}$

$\vec{q}=\hat{i}+\hat{j}+\hat{7}k$

$\vec{p}\times\vec{q}=\begin{vmatrix}i & j & k\\ 1 & 4 & 22\\ 1 & 1 & 7\end{vmatrix}=6\hat{i}+15\hat{j}-3\hat{k}$

S.D. $=\frac{|\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})|}{|\vec{p}\times\vec{q}|}=\frac{|36+225+9|}{\sqrt{36+225+9}}=3\sqrt{30}$

22. If volume of parallelepiped whose there coterminous edges are $\vec{u}=\hat{i}+\hat{j}+\lambda\hat{k}, \vec{v}=2\hat{i}+\hat{j}+\hat{k}$ & $\vec{w}=\hat{i}+\hat{j}+3\hat{k}$ is 1 cubic unit then cosine of angle between $\hat{u}$ and $\hat{v}$ is

A) $\frac{7}{3\sqrt{10}}$ B) $\frac{7}{6\sqrt{3}}$ C) $\frac{5}{3\sqrt{3}}$ D) $\frac{5}{7}$

Ans. B

Sol. $\pm 1=\begin{vmatrix}1 & 1 & \lambda\\ 1 & 1 & 3\\ 2 & 1 & 1\end{vmatrix}\Rightarrow=-\lambda+3=\pm 1\Rightarrow\lambda=2$ or $\lambda=4$

For $\lambda=4$

$\cos\theta=\frac{2+1+4}{\sqrt{6}\sqrt{18}}=\frac{7}{6\sqrt{3}}$

23. Find the sum $\sum_{k=1}^{20}(1+2+3+\ldots+k)$

Ans. 1540

Sol. $\sum_{k=1}^{20}\frac{k(k+1)}{2}=\frac{1}{2}\sum_{k=1}^{20}(k^2+k)$

$=\frac{1}{2}\left[\frac{20(21)(41)}{6}+\frac{20(21)}{2}\right]$

$=\frac{1}{2}\left[\frac{420\times 41}{6}+\frac{20\times 21}{2}\right]$

$=\frac{1}{2}[2870+210]=1540$

24. If normal at P on the curve $y^2 - 3x^2 + y + 10 = 0$ passes through the point $(0, 3/2)$ then slope of tangent at P is n. The value of $|n|$ is equal to

Ans. 4

Sol. $P \equiv (x_1, y_1)$

$$2yy' - 6x + y' = 0 \Rightarrow y' = \left(\frac{6x_1}{1 + 2y_1}\right)$$

$$\left(\frac{\frac{3}{2} - y_1}{-x_1}\right) = -\left(\frac{1 + 2y_1}{6x_1}\right)$$

$9 - 6y_1 = 1 + 2y_1$

$\Rightarrow y_1 = 1$

$\therefore x_1 = \pm 2$

$\therefore$ Slope of tangent $= \left(\frac{\pm 12}{3}\right)$

$= \pm 4$

$\therefore |n| = 4$

25. If $2x^2 + (a - 10)x + \frac{33}{2} = 2a, a \in Z^+$ has real roots, then minimum value of 'a' is equal to

Ans. 8

Sol. $D \geq 0$

$$(a - 10)^2 - 4(2)\left(\frac{33}{2} - 2a\right) \geq 0$$

$(a - 10)^2 - 4(33 - 4a) \geq 0$

$a^2 - 4a - 32 \geq 0 \Rightarrow a \in (-\infty, -4] \cup [8, \infty)$