Avanti

JEE Mains 2020

8th Jan Shift 2 Paper – Mathematics

1. Let $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c}$ is nonzero vector and $\vec{b} \times \vec{c} = \vec{b} \times \vec{a}, \vec{a} \cdot \vec{c} = 0$, find $\vec{b} \cdot \vec{c}$

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $-\frac{1}{2}$ D) $-\frac{1}{3}$

Ans. C

Sol. $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times (\vec{b} \times \vec{a})$

$-(\vec{a} \cdot \vec{b})\, \vec{c} = (\vec{a} \cdot \vec{a})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{a}$

$-4\vec{c} = 6(\hat{\imath} - \hat{\jmath} + \hat{k}) - 4(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

$-4\vec{c} = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$$\vec{c} = -\frac{1}{2}(\hat{\imath} + \hat{\jmath} + \hat{k})$$

$$\vec{b} \cdot \vec{c} = -\frac{1}{2}$$

2. Let coefficient of x^4 and x^2 in the expansion of $\left(x + \sqrt{x^2 - 1}\right)^6 + \left(x - \sqrt{x^2 - 1}\right)^6$ is α and β then $\alpha - \beta$ is equal to

A) 48 B) 60 C) -132 D) -60

Ans. C

Sol. $2[{}^6C_0 x^6 + {}^6C_2 x^4(x^2 - 1) + {}^6C_4 x^2(x^2 - 1)^2 + {}^6C_6(x^2 - 1)^3]$

$= 2[x^6 + 15(x^6 - x^4) + 15x^2(x^4 - 2x^2 + 1) + (-1 + 3x^2 - 3x^4 + x^6)]$

$= 2(32x^6 - 48x^4 + 18x^2 - 1)$

$\alpha = -96\ \&\ \beta = 36 \therefore \alpha - \beta = -132$

3. Differential equation of $x^2 = 4b(y + b)$, where be is a parameter, is

A) $x\left(\frac{dy}{dx}\right)^2 = 2y\frac{dy}{dx} + x^2$ B) $x\left(\frac{dy}{dx}\right)^2 = 2y\frac{dy}{dx} + x$

C) $x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx} + x^2$ D) $x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx} + 2x^2$

Ans. B

Sol. $2x = 4by' \Rightarrow b = \frac{x}{2y'}$

So differential equation is $x^2 = \frac{2x}{y'} \cdot y + \left(\frac{x}{y'}\right)^2 \Rightarrow x\left(\frac{dy}{dx}\right)^2 = 2y\frac{dy}{dx} + x$

4. Image of $(1, 2, 3)$ w.r.t. a plane is $\left(\frac{-7}{3}, \frac{-4}{3}, \frac{-1}{3}\right)$ then which of the following points lie on the plane

A) $(-1, 1, -1)$ B) $(-1, -1, -1)$ C) $(-1, -1, 1)$ D) $(1, 1, -1)$

Ans. D

Sol. d.r. of normal to the plane

$\frac{10}{3}, \frac{10}{3}, \frac{10}{3},$

$1, \quad 1, \quad 1,$

midpoint of P and Q is $\left(\frac{-2}{3}, \frac{1}{3}, \frac{4}{3}\right)$

equation of plane $x + y + z = 1$

5. $\lim_{x \to 0} \frac{\int_0^x t \sin(10t)\, dt}{x}$ is equal to

A) 1 B) 10 C) 5 D) 0

Ans. D

Sol. Using L' Hospital

$$\lim_{x \to 0} \frac{x \sin(10x)}{1} = 0$$

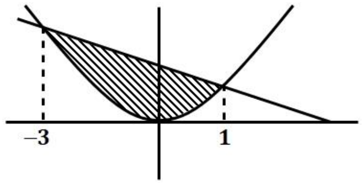
6. Let P be the set of points (x,y) such that $x^2 \le y \le -2x+3$. Then area of region bounded by points in set P is-

A) $\frac{16}{3}$ B) $\frac{32}{3}$ C) $\frac{29}{3}$ D) $\frac{20}{3}$

Ans. B

Sol. Point of intersection of $y = x^2$ & $y = -2x+3$ is

Obtained by, $x^2 + 2x - 3 = 0 \Rightarrow x = -3, 1$



$$\text{So, area} = \int_{-3}^{1} (3 - 2x - x^2)\,dx = 3(4) - 2\left(\frac{1^2 - 3^2}{2}\right) - \left(\frac{1^3 + 3^3}{3}\right)$$

$$= 12 + 8 - \frac{28}{3} = \frac{32}{3}$$

7. Let $f(x) = \frac{d[x]}{x^2+1} : (1,3) \to R$ then range of $f(x)$ is (where $[\cdot]$ denotes greatest integer function)

A) $\left(0,\frac{1}{2}\right) \cup \left(\frac{3}{5},\frac{5}{7}\right]$ B) $\left(\frac{2}{5},\frac{1}{2}\right) \cup \left(\frac{3}{5},\frac{4}{5}\right]$ C) $\left(\frac{2}{5},1\right) \cup \left(1,\frac{4}{5}\right]$ D) $\left(0,\frac{1}{3}\right) \cup \left(\frac{2}{5},\frac{4}{5}\right]$

Ans. B

Sol. $f(x)\begin{cases} \frac{x}{x^2+1}; & x \in (1,2) \\ \frac{2x}{x^2+1}; & x \in [2,3) \end{cases}$

$\therefore f(x)$ is a decreasing function

$$\therefore y \in \left(\frac{2}{5},\frac{1}{2}\right) \cup \left(\frac{6}{10},\frac{4}{5}\right] \Rightarrow y \in \left(\frac{2}{5},\frac{1}{2}\right) \cup \left(\frac{3}{5},\frac{4}{5}\right]$$

8. Let $A = \begin{bmatrix} 2 & 2 \\ 9 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then value of $10A^{-1}$ is-

A) $4I - A$ B) $6I - A$ C) $A - 4I$ D) $A - 6I$

Ans. D

Sol. Characteristics equation of matrix 'A' is

$$\begin{vmatrix} 2-x & 2 \\ 9 & 4-x \end{vmatrix} = 0 \Rightarrow x^2 - 6x - 10 = 0$$

$\therefore A^2 - 6A - 10I = 0$

$\Rightarrow 10A^{-1} = A - 6I$

9. Solution set of $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$ contains

A) singleton set B) two elements

C) at least four elements D) infinite elements

Ans. A

Sol. Let $3^x = t$

$t(t-1) + 2 = |t-1| + |t-2|$

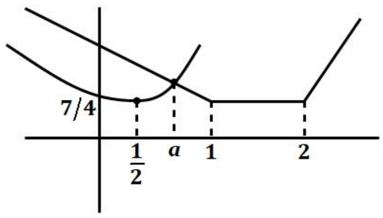
$t^2 - t + 2 = |t-1| + |t-2|$

are positive solution

$t = a$

$3^x = a$

$x = \log_3 a$ so singleton set



10. Mean and variance of 20 observation are 10 and 4. It was found, that in place of 11, 9 was taken by mistake find correct variance.

A) 3.99 B) 3.98 C) 4.01 D) 4.02

Ans. A

Sol. $\frac{\sum x_i}{20} = 10$ (1)

$\frac{\sum x_i^2}{20} - 100 = 4$ (2)

$\sum x_i^2 = 104 \times 20 = 2080$

$$\text{Actual mean} = \frac{200 - 9 + 11}{20} = \frac{202}{20}$$

$$\text{Variance} = \frac{2080 - 81 + 121}{20} - \left(\frac{202}{20}\right)^2$$

$$= \frac{2120}{20} - (10.1)^2 = 106 - 102.01 = 3.99$$

11. $\begin{array}{l} \lambda x + 2y + 2z = 5 \\ 2\lambda x + 3y + 5z = 8 \\ 4x + \lambda y + 6z = 10 \end{array}$ for the system of equation check the correct option.

A) Infinite solutions when $\lambda = 8$
B) Infinite solutions when $\lambda = 2$
C) No solutions when $\lambda = 8$
D) No solutions when $\lambda = 2$

Ans. D

Sol. $D = \begin{vmatrix} \lambda & 2 & 2 \\ 2\lambda & 3 & 5 \\ 4 & \lambda & 6 \end{vmatrix}$

$D = (\lambda + 8)(2 - \lambda)$

for $\lambda = 2$

$D_1 = \begin{vmatrix} 5 & 2 & 2 \\ 8 & 3 & 5 \\ 10 & 2 & 6 \end{vmatrix}$

$= 5[18 - 10] - 2[48 - 50] + 2(16 - 30]$

$= 40 + 4 - 28 \neq 0$

No solutions for $\lambda = 2$

12. For an A.P. $T_{10} = \frac{1}{20}; T_{20} = \frac{1}{20}$. Find sum of first 200 term.

A) $201\frac{1}{2}$
B) $101\frac{1}{2}$
C) $301\frac{1}{2}$
D) $100\frac{1}{2}$

Ans. D

Sol. $T_{10} = \frac{1}{20} = a + 9d$ (1)

$T_{20} = \frac{1}{10} = a + 19d$ (2)

$\Rightarrow a = \frac{1}{200}, d = \frac{1}{200} \Rightarrow S_{200}$

$= \frac{200}{2}\left[\frac{2}{200} + \frac{199}{200}\right] = \frac{201}{2} = 100\frac{1}{2}$

13. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$ and $a = (1+\alpha)\sum_{k=0}^{100} \alpha^{2k}, b = \sum_{k=0}^{100} \alpha^{3k}$. If a and b are roots of quadratic equation then quadratic equation is

A) $x^2 - 102x + 101 = 0$
B) $x^2 - 101x + 100 = 0$
C) $x^2 + 101x + 100 = 0$
D) $x^2 + 102x + 100 = 0$

Ans. A

Sol. $\alpha = \omega, b = 1 + \omega^2 + \omega^6 + \cdots = 101$

$a = (1 + \omega)(1 + \omega^2 + \omega^4 + \cdots \omega^{198} + \omega^{200})$

$= (1 + \omega)\frac{(1 - (\omega^2)^{101})}{1 - \omega^2} = \frac{(1 + \omega)(1 - \omega)}{1 - \omega^2} = 1$

Equation $x^2 - (101 + 1)x + (101) \times 1 = 0$

$\Rightarrow x^2 - 102x + 101 = 0$

14. Let $f(x)$ is a three degree polynomial for which $f'(1) = 0, f''(1) = 0, f(-1) = 10, f(1) = 6$ then local minima of $f(x)$ exist at

A) $x = 3$
B) $x = 2$
C) $x = 1$
D) $x = -1$

Ans. A

Sol. Let $f(x) = ax^3 + bx^2 + cx + c$

$a = \frac{1}{4} \quad d = \frac{35}{4}$

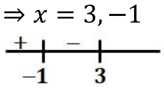
$b = \frac{-3}{4} \quad c = -\frac{9}{4}$

$\Rightarrow f(x) = a(x^3 - 3x^2 - 9x) + d$

$f'(x) = \frac{3}{4}(x^2 - 2x - 3)$

$\Rightarrow f'(x) = 0$

$\Rightarrow x = 3, -1$



local minima exist at $x = 3$

15. Let A and B are two events such that $P(\text{exactly one}) = \frac{2}{5}, P(A \cup B) = \frac{1}{2}$ then $P(A \cap B) =$

A) $\frac{1}{10}$ B) $\frac{2}{9}$ C) $\frac{1}{8}$ D) $\frac{1}{12}$

Ans. A

Sol. $P(\text{exactly one}) = \frac{2}{5}$

$P(A) + P(B) - 2P(A \cap B) = \frac{2}{5}$

$P(A \cup B) = \frac{1}{2}$

$\Rightarrow P(A) + P(B) - P(A \cap B) = \frac{1}{2}$

$\therefore P(A \cap B) = \frac{1}{2} - \frac{2}{5} = \frac{5-4}{10} = \frac{1}{10}$

16. Let $I = \int_1^2 \frac{1dx}{\sqrt{2x^3-9x^2+12x+4}}$ then

A) $\frac{1}{9} < I^2 < \frac{1}{8}$ B) $\frac{1}{2} < I^2 < \frac{1}{2}$

C) $\frac{1}{9} < I < \frac{1}{8}$ D) $\frac{1}{3} < I < \frac{1}{2}$

Ans. A

Sol. $f(x) = \frac{1}{\sqrt{2x^3-9x^2+12x+4}}$

$f'(x) = \frac{-1}{2} \frac{(6x^2 - 18x + 12)}{(2x^3 - 9x^2 + 12x + 4)^{\frac{3}{2}}}$

$= \frac{-6(x-1)(x-2)}{2(2x^3 - 9x^2 + 12x + 4)^{\frac{3}{2}}}$

$f(1) = \frac{1}{3},$

$f(2) = \frac{1}{\sqrt{8}}$

$\frac{1}{3} < I < \frac{1}{\sqrt{8}}$

17. Normal at (2,2) to curve $x^2 + 2xy - 3y^2 = 0$ is L. Then perpendicular distance from origin to line L is

A) $4\sqrt{2}$ B) 2 C) $2\sqrt{2}$ D) 4

Ans. C

Sol. $x^2 + 2xy - 3y^2 = 0$

$x^2 + 3xy - xy - 3y^2 = 0$

$x^2 + 3xy - xy - 3y^2 = 0$

$(x-y)(x+3y) = 0$

$x - y = 0$

$x + 3y = 0$

$(2,2)$ satisfy $x - y = 0$

Normal:

$x + y = \lambda$

$\lambda = 4$

Hence $x + y = 4$

Perpendicular distance from origin

$= \left|\frac{0+0-4}{\sqrt{2}}\right| = 2\sqrt{2}$

18. Which of the following is tautology-

A) $\sim (p \vee \sim q) \to (p \vee q)$ B) $(\sim p \vee q) \to (p \vee q)$

C) $\sim (\wedge \sim q) \to (p \vee q)$ D) $\sim (p \vee \sim q) \to (p \wedge q)$

Ans. A

Sol. $(\sim p \wedge q) \to (P \vee q)$

$\sim \{(\sim p \wedge q) \wedge (\sim p \wedge \sim q)\}$

$\sim \{\sim P \wedge f\}$

19. If a hyperbola has vertices $(\pm 6, 0)$ and $P(10, 16)$ lies on it. Then the equation of normal at P is

A) $2x + 5y = 100$

B) $2x + 5y = 10$

C) $2x - 5y = 100$

D) $5x + 2y = 100$

Ans. A

Sol. Vertex is at $(\pm 6, 0)$

$\therefore a = 6$

Let the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Putting point $P(10, 16)$ on the hyperbola

$$\frac{100}{36} - \frac{256}{b^2} = 1 \Rightarrow b^2 = 144$$

$\therefore$ hyperbola is $\frac{x^2}{36} - \frac{y^2}{144} = 1$

$\therefore$ equation of normal is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

$\therefore$ putting we get $2x + 5y = 100$

20. If $y = mx + c$ is a tangent to the circle $(x-3)^2 + y^2 = 1$ and also the perpendicular to the tangent to the circle $x^2 + y^2 = 1$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then

A) $c^2 + 6c + 7 = 0$

B) $c^2 - 6c + 7 = 0$

C) $c^2 + 6c - 7 = 0$

D) $c^2 - 6c - 7 = 0$

Ans. A

Sol. Slope of tangent to $x^2 + y^2 = 1$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$x^2 + y^2 = 1$

$2x + 2yy' = 0$

$$y' = -\frac{x}{y} = -1$$

$y = mx + c$ is tangent of $x^2 + y^2 = 1$

so $m = 1$

$y = x + c$

now distance of $(3, 0)$ from $y = x + c$ is

$$\left|\frac{c+3}{\sqrt{2}}\right| = 1$$

$c^2 + 6c + 9 = 2$

$c^2 + 6c + 7 = 0$

21. Let $\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \frac{1}{7}$ and $\sqrt{\frac{1-\cos 2\beta}{2}} = \frac{1}{\sqrt{10}}$ where $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. Then the $(\alpha + 2\beta)$ is equal to

Ans. 1

Sol. $\frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha} = \frac{1}{7}$ and $\frac{\sqrt{2}\sin\beta}{\sqrt{2}} = \frac{1}{\sqrt{10}}$

$$\tan\alpha = \frac{1}{7}$$

$$\sin\beta = \frac{1}{\sqrt{10}}$$

$$\tan\beta = \frac{1}{3}$$

$$\tan 2\beta = \frac{2 \cdot \frac{1}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$$

$$\tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha \tan 2\beta}$$

$$= \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \cdot \frac{3}{4}} = \frac{\frac{4+21}{28}}{\frac{25}{28}} = 1$$

22. The number of four letter words that can be made from the letters of word "EXAMINATION" is

Ans. 2454

Sol. EXAMINATION

$2N, 2A, 2I, E, X, M, T, O$

Case I: All are different so ${}^{8}P_4 = \frac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

Case II: 2 same and 2 different so ${}^{3}C_1 \cdot {}^{7}C_2 \cdot \frac{4!}{2!} = 3 \cdot 21 \cdot 12 = 756$

Case III: 2 same and 2 same so ${}^{3}C_2 \cdot \frac{4!}{2!2!} = 3 \cdot 6 = 18$

$\therefore$ Total $= 1680 + 756 + 18 = 2454$

23. Let the line $y = mx$ intersects the curve $y^2 = x$ at P and tangent to $y^2 = x$ at P intersects x-axis at Q. If area $(\Delta OPQ) = 4$, find $m (m > 0)$

Ans. 0.5

Sol. $2ty = x + t^2$

$Q(-t^2, 0)$

$$\frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ t^2 & t & 1 \\ -t^2 & 0 & 1 \end{vmatrix}\right| = 4$$

$|t|^3 = 8$

$t = \pm 2 (t > 0)$

$m = \frac{1}{2}$

24. $\sum_{n=1}^{7} \frac{n(n+1)(2n+1)}{4}$ is equal to

Ans. 504

Sol. $\frac{1}{4}[\sum_{n=1}^{7}(2n^3 + 3n^2 + n)]$

$$\frac{1}{4}\left[2\left(\frac{7 \cdot 8}{2}\right)^2 + 3\left(\frac{7 \cdot 8 \cdot 15}{6}\right) + 7 \cdot \frac{8}{2}\right]$$

$$\frac{1}{4}[2 \times 49 \times 16 + 28 \times 15 + 28\,]$$

$$\frac{1}{4}[1568 + 420 + 28] = 504$$